Fourier series

Frames 1 to 89

Learning outcomes

When you have completed this Programme you will be able to:

- Determine the period and amplitude of a periodic function
- Write down the harmonics of a periodic trigonometric function
- Give an analytic description of a non-sinusoidal periodic function
- Evaluate integrals with periodic integrands
- Demonstrate the orthogonality of the trigonometric functions $\sin nx$ and $\cos nx$ for $n = 0, 1, 2, \ldots$
- Describe a periodic function as a Fourier series subject to Dirichlet conditions
- Obtain the Fourier coefficients and hence the Fourier series of a periodic function
- Describe the effects of the harmonics in the construction of the Fourier series
- Find the value of the Fourier series at a point of discontinuity of the periodic function
- Derive the Fourier series of non-sinusoidal periodic functions
- Recognise even and odd functions and their products
- Derive the Fourier sine and cosine series for odd and even functions respectively
- Derive half-range Fourier series
- Recognise the condition for the Fourier series to contain only odd or only even harmonics
- Explain the significance of the term $a_0/2$

Prerequisite: Engineering Mathematics (Fifth Edition)
Programmes 15 Integration 1 and 17 Reduction formulas

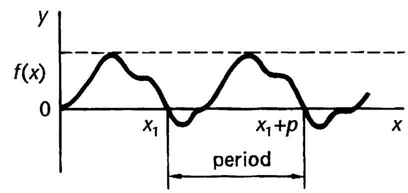
Introduction

1

We have seen earlier that many functions can be expressed in the form of infinite series. Problems involving various forms of oscillations are common in fields of modern technology and *Fourier series*, with which we shall now be concerned, enable us to represent a periodic function as an infinite trigonometrical series in sine and cosine terms. One important advantage of a Fourier series is that it can represent a function containing discontinuities, whereas Maclaurin's and Taylor's series require the function to be continuous throughout.

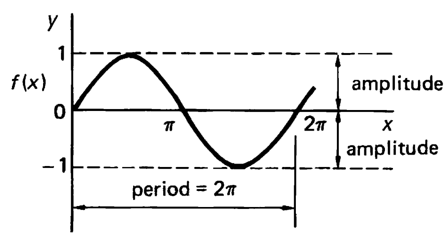
Periodic functions

A function $f(x)$ is said to be *periodic* if its function values repeat at regular intervals of the independent variable. The regular interval between repetitions is the *period* of the oscillations.

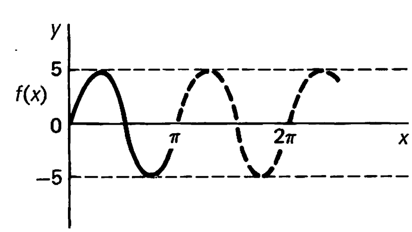


Graphs of $y = A\sin nx$

(a) $y = \sin x$
The obvious example of a periodic function is $y = \sin x$, which goes through its complete range of values while x increases from 0° to 360°. The period is therefore 360° or 2π radians and the amplitude, the maximum displacement from the position of rest, is 1.



(b) $y = 5\sin 2x$
The amplitude is 5.
The period is 180° and there are thus 2 complete cycles in 360°.



▶

(c) $y = A\sin nx$
Thinking along the same lines, the function $y = A\sin nx$ has amplitude ....; period ....;
and will have .... complete cycles in 360°.

2

$$\text{amplitude} = A;\ \text{period} = \frac{360°}{n} = \frac{2\pi}{n};\ n \text{ cycles in } 360°$$

Graphs of $y = A\cos nx$ have the same characteristics.

By way of revising earlier work, then, complete the following short exercise.

Exercise

In each of the following, state (a) the amplitude and (b) the period.

1 $y = 3\sin 5x$
2 $y = 2\cos 3x$
3 $y = \sin\frac{x}{2}$
4 $y = 4\sin 2x$
5 $y = 5\cos 4x$
6 $y = 2\sin x$
7 $y = 3\cos 6x$
8 $y = 6\sin\frac{2x}{3}$

Deal with all eight. They will not take much time.

3

No.	Amplitude	Period	No.	Amplitude	Period
1	3	$2\pi/5$	**5**	5	$\pi/2$
2	2	$2\pi/3$	**6**	2	π
3	1	4π	**7**	3	$\pi/3$
4	4	π	**8**	6	3π

Harmonics

A function $f(x)$ is sometimes expressed as a series of a number of different sine components. The component with the largest period is the *first harmonic*, or *fundamental* of $f(x)$.

$y = A_1 \sin x$ is the first harmonic or fundamental
$y = A_2 \sin 2x$ is the second harmonic
$y = A_3 \sin 3x$ is the third harmonic, etc.

and in general
$y = A_n \sin nx$ is the .... harmonic, with amplitude .... and period ....

$$\boxed{n\text{th harmonic; amplitude } A_n\text{; period} = \frac{2\pi}{n}}$$

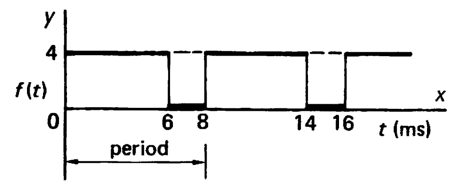
Non-sinusoidal periodic functions

Although we introduced the concept of a periodic function via a sine curve, a function can be periodic without being obviously sinusoidal in appearance.

Example

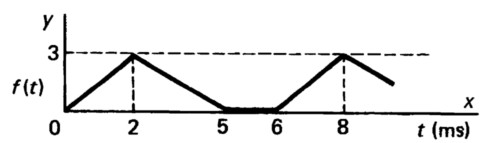
In the following cases, the x-axis carries a scale of t in milliseconds.

(a)



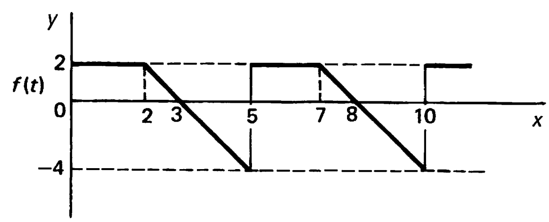
period = 8 ms

(b)



period = ...

(c)



period = ...

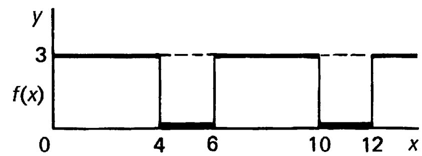
5

$$\boxed{\text{(b) period} = 6 \text{ ms; (c) period} = 5 \text{ ms}}$$

Analytic description of a periodic function

A periodic function can be defined analytically in many cases.

Example 1



(a) Between $x = 0$ and $x = 4$, $y = 3$, i.e. $f(x) = 3 \quad 0 < x < 4$

(b) Between $x = 4$ and $x = 6$, $y = 0$, i.e. $f(x) = 0 \quad 4 < x < 6$

So we could define the function by

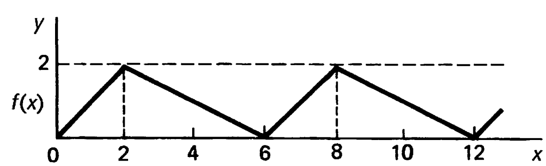
$$f(x) = \begin{cases} 3 & 0 < x < 4 \\ 0 & 4 < x < 6 \end{cases}$$
$$f(x+6) = f(x)$$

the last line indicating that ....

6

$$\boxed{\text{the function is periodic with period 6 units}}$$

Example 2



In this case

(a) Between $x = 0$ and $x = 2$, $y = x$ i.e. $f(x) = x \quad 0 < x < 2$

(b) Between $x = 2$ and $x = 6$, $y = -\dfrac{x}{2} + 3$, i.e. $f(x) = 3 - \dfrac{x}{2} \quad 2 < x < 6$

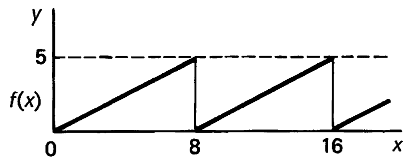
(c) The period is 6 units i.e. $f(x+6) = f(x)$.

So we have

$$f(x) = \begin{cases} x & 0 < x < 2 \\ 3 - \dfrac{x}{2} & 2 < x < 6 \end{cases}$$
$$f(x+6) = f(x).$$

▶

Example 3



In this case ....
....

7

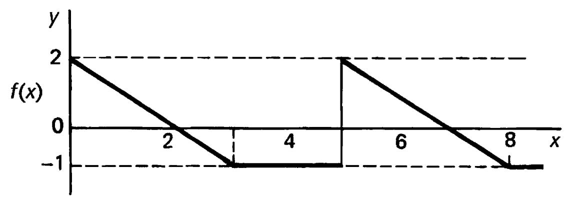
$$f(x) = \frac{5x}{8} \qquad 0 < x < 8$$
$$f(x+8) = f(x)$$

Here is a short exercise.

Exercise

Define analytically the periodic functions shown.

1



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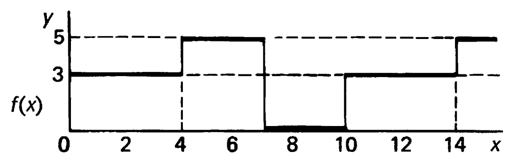

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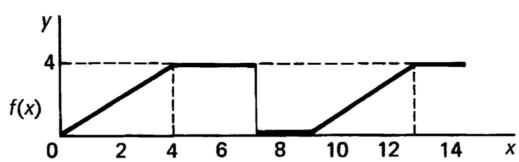

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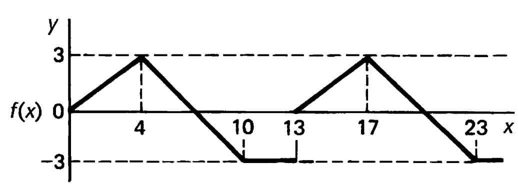

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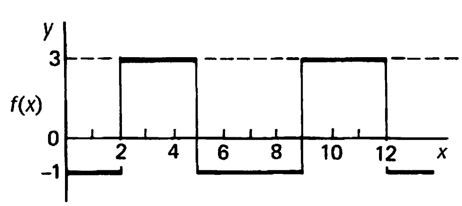


Finish all five and then check the results.

8

Here are the details.

1
$$f(x) = \begin{cases} 2 - x & 0 < x < 3 \\ -1 & 3 < x < 5 \end{cases}$$
$f(x+5) = f(x)$.

2
$$f(x) = \begin{cases} 3 & 0 < x < 4 \\ 5 & 4 < x < 7 \\ 0 & 7 < x < 10 \end{cases}$$
$f(x+10) = f(x)$.

3
$$f(x) = \begin{cases} x & 0 < x < 4 \\ 4 & 4 < x < 7 \\ 0 & 7 < x < 9 \end{cases}$$
$f(x+9) = f(x)$.

4
$$f(x) = \begin{cases} \dfrac{3x}{4} & 0 < x < 4 \\ 7 - x & 4 < x < 10 \\ -3 & 10 < x < 13 \end{cases}$$
$f(x+13) = f(x)$.

5
$$f(x) = \begin{cases} -1 & 0 < x < 2 \\ 3 & 2 < x < 5 \\ -1 & 5 < x < 7 \end{cases}$$
$f(x+7) = f(x)$.

Now we have the same thing in reverse.

Exercise

Sketch the graphs of the following, inserting relevant values.

1
$$f(x) = \begin{cases} 4 & 0 < x < 5 \\ 0 & 5 < x < 8 \end{cases}$$
$f(x+8) = f(x)$.

2
$$f(x) = 3x - x^2 \qquad 0 < x < 3$$
$f(x+3) = f(x)$.

3
$$f(x) = \begin{cases} 2\sin x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$$
$f(x+2\pi) = f(x)$.

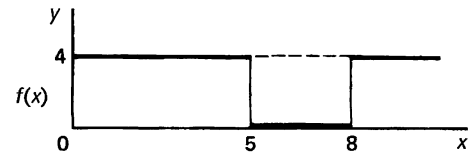
4
$$f(x) = \begin{cases} \dfrac{x}{2} & 0 < x < \pi \\ \pi - \dfrac{x}{2} & \pi < x < 2\pi \end{cases}$$
$f(x+2\pi) = f(x)$.

5
$$f(x) = \begin{cases} \dfrac{x^2}{4} & 0 < x < 4 \\ 4 & 4 < x < 6 \\ 0 & 6 < x < 8 \end{cases}$$
$f(x+8) = f(x)$.

9

Here they are: check carefully.

1



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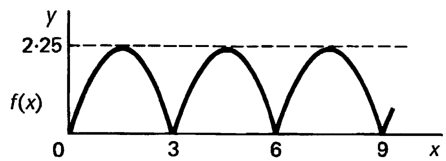

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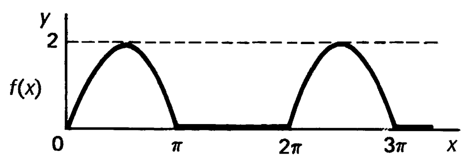

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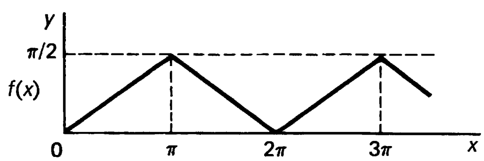

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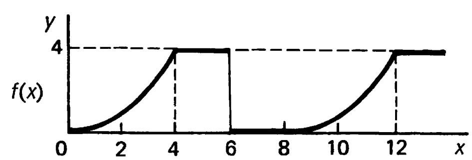


All this is in preparation for what is to come, so let us now consider Fourier series.

Move on then to the next frame

Integrals of periodic functions

10

Before we proceed we need to consider some specific integrals involving integers m and n. These are integrals over a single period of periodic integrands. You will already know some of these and the others you will easily be able to work out. The integrals that we are concerned with are those of sines, cosines and their combinations where the integration is over a single period from $-\pi$ to π. First, though, we list the integral of the unit constant over the period.

▶

1 $\displaystyle\int_{-\pi}^{\pi} dx = \Big[x\Big]_{-\pi}^{\pi} = 2\pi$

2 $\displaystyle\int_{-\pi}^{\pi} \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} \qquad (n \neq 0)$

$$= \frac{\sin n\pi}{n} - \frac{\sin(-n\pi)}{n}$$

$$= 0 \quad \text{because } \sin n\pi = 0$$

3 $\displaystyle\int_{-\pi}^{\pi} \sin nx\,dx = \ldots\ldots\ldots\ldots \qquad (n \neq 0)$

11

$$\boxed{\int_{-\pi}^{\pi} \sin nx\,dx = 0}$$

Because

$$\int_{-\pi}^{\pi} \sin nx\,dx = \left[-\frac{\cos nx}{n}\right]_{-\pi}^{\pi} \qquad (n \neq 0)$$

$$= -\frac{\cos n\pi}{n} + \frac{\cos(-n\pi)}{n}$$

$$= 0 \quad \text{because } \cos(-x) = \cos x$$

4 $\displaystyle\int_{-\pi}^{\pi} \cos^2 nx\,dx = \int_{-\pi}^{\pi} \frac{\cos 2nx + 1}{2}\,dx \quad \text{because } \cos 2A = 2\cos^2 A - 1$

$$= \left[\frac{\sin 2nx}{4n} + \frac{x}{2}\right]_{-\pi}^{\pi} \qquad (n \neq 0)$$

$$= \frac{\sin 2n\pi}{4n} + \frac{\pi}{2} - \frac{\sin(-2n\pi)}{4n} - \frac{(-\pi)}{2}$$

$$= \pi$$

5 $\displaystyle\int_{-\pi}^{\pi} \sin^2 nx\,dx = \ldots\ldots\ldots\ldots \qquad (n \neq 0)$

12

$$\boxed{\int_{-\pi}^{\pi} \sin^2 nx\,dx = \pi}$$

Because

$$\int_{-\pi}^{\pi} \sin^2 nx\,dx = \int_{-\pi}^{\pi} \frac{1 - \cos 2nx}{2}\,dx \quad \text{because } \cos 2A = 1 - 2\sin^2 A$$

$$= \left[\frac{x}{2} - \frac{\sin 2nx}{4n}\right]_{-\pi}^{\pi} \qquad (n \neq 0)$$

$$= \frac{\pi}{2} - \frac{\sin 2n\pi}{4n} - \frac{(-\pi)}{2} + \frac{\sin(-2n\pi)}{4n}$$

$$= \pi$$

▶

6 $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \, dx$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x] \, dx$$

because $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$$= \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} \qquad (m \neq n)$$

$$= \frac{\sin(m+n)\pi}{m+n} + \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)(-\pi)}{m+n} - \frac{\sin(m-n)(-\pi)}{m-n}$$

$$= 0$$

7 $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \dots\dots\dots\dots \qquad (m \neq n)$

13

$$\boxed{\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad m \neq n}$$

Because

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x] \, dx$$

because $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$$= \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} \qquad (m \neq n)$$

$$= \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)\pi}{m+n} - \frac{\sin(m-n)(-\pi)}{m-n} + \frac{\sin(m+n)(-\pi)}{m+n}$$

$$= 0$$

8 $\displaystyle\int_{-\pi}^{\pi} \cos mx \sin nx \, dx \qquad (m \neq n)$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\sin(m+n)x - \sin(m-n)x] \, dx$$

because $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$$= \frac{1}{2}\left[-\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right]_{-\pi}^{\pi} \qquad (m \neq n)$$

$$= \frac{1}{2}\left(-\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n} + \frac{\cos(m+n)(-\pi)}{m+n} - \frac{\cos(m-n)(-\pi)}{m-n}\right)$$

$= 0$ because $\cos(-x) = \cos x$

And finally, when $m = n$

9 $\displaystyle\int_{-\pi}^{\pi} \cos mx \sin mx \, dx = \dots\dots\dots\dots$

14

$$\boxed{\int_{-\pi}^{\pi} \cos mx \sin mx \, dx = 0}$$

Because

$$\begin{aligned}
&\int_{-\pi}^{\pi} \cos mx \sin mx \, dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \sin 2mx \, dx \quad \text{because } \sin 2A = 2\sin A\cos A \\
&= \frac{1}{2}\left[-\frac{\cos 2mx}{2m}\right]_{-\pi}^{\pi} \qquad (m \neq 0) \\
&= \frac{1}{2}\left(-\frac{\cos 2m\pi}{2m} + \frac{\cos 2m(-\pi)}{2m}\right) \\
&= 0 \quad \text{because } \cos(-x) = \cos x
\end{aligned}$$

15

Summary

1 $\int_{-\pi}^{\pi} dx = [x]_{-\pi}^{\pi} = 2\pi$

2 $\int_{-\pi}^{\pi} \cos nx \, dx = 0$

3 $\int_{-\pi}^{\pi} \sin nx \, dx = 0$

4 $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \pi\delta_{mn}$ where $\delta_{mn} = \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}$

(δ_{mn} is called the Kronecker delta)

5 $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \pi\delta_{mn}$

6 $\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0$

Note that the same results are obtained no matter what the end points of the integrals are, *provided that the interval between them is one period.* So, for example

$$\begin{aligned}
\int_{k}^{k+2\pi} \cos nx \, dx &= \left[\frac{\sin nx}{n}\right]_{k}^{k+2\pi} \qquad (n \neq 0) \\
&= \frac{\sin(nk + 2n\pi)}{n} - \frac{\sin nk}{n} \\
&= 0 \quad \text{because } \sin(x + 2n\pi) = \sin x
\end{aligned}$$

Now to put all these integrals to practical use

Orthogonal functions

16

If two different functions $f(x)$ and $g(x)$ are defined on the interval $a \le x \le b$ and

$$\int_a^b f(x)g(x)\,dx = 0$$

then we say that the two functions are **orthogonal** to each other on the interval $a \le x \le b$. In the previous frames we have seen that the trigonometric functions $\sin nx$ and $\cos nx$ where $n = 0, 1, 2, \ldots$ form an infinite collection of periodic functions that are mutually orthogonal on the interval $-\pi \le x \le \pi$, indeed on any interval of width 2π. That is

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = 0 \quad \text{for } m \ne n$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0 \quad \text{for } m \ne n$$

and

$$\int_{-\pi}^{\pi} \cos mx \sin nx\,dx = 0$$

Fourier series

17

Given that certain conditions are satisfied then it is possible to write a periodic function of period 2π as a series expansion of the orthogonal periodic functions just discussed. That is, if $f(x)$ is defined on the interval $-\pi \le x \le \pi$ where $f(x + 2n\pi) = f(x)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

This is the **Fourier series** expansion of $f(x)$ where the a_n and b_n are constants called the *Fourier coefficients*. But how do we find the values of these constants? Quite easily. We make use of the mutual orthogonality of the trigonometric functions in the expansion.

▶

For example, to find a_{10} we multiply $f(x)$ by $\cos 10x$ and integrate over a period. That is

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\cos 10x\,dx \\
&= \int_{-\pi}^{\pi}\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\right)\cos 10x\,dx \\
&= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos 10x\,dx+\sum_{n=1}^{\infty}a_n\int_{-\pi}^{\pi}\cos nx\cos 10x\,dx \\
&\qquad +\sum_{n=1}^{\infty}b_n\int_{-\pi}^{\pi}\sin nx\cos 10x\,dx \\
&= \frac{a_0}{2}\times 0+\sum_{n=1}^{\infty}a_n\pi\delta_{n,10}+\sum_{n=1}^{\infty}b_n\times 0 \\
&= a_0\pi\times 0+a_1\pi\times 0+\ldots+a_9\pi\times 0+a_{10}\pi\times 1+a_{11}\pi\times 0+\ldots \\
&= a_{10}\pi
\end{aligned}$$

So that

$$a_{10}=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 10x\,dx$$

In just the same way $\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \ldots\ldots\ldots\ldots$

18

$$\boxed{\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_m\pi}$$

Because

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\cos mx\,dx \\
&= \int_{-\pi}^{\pi}\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\right)\cos mx\,dx \\
&= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos mx\,dx+\sum_{n=1}^{\infty}a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx \\
&\qquad +\sum_{n=1}^{\infty}b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx \\
&= \frac{a_0}{2}\times 0+\sum_{n=1}^{\infty}a_n\pi\delta_{n,m}+\sum_{n=1}^{\infty}b_n\times 0 \\
&= a_m\pi
\end{aligned}$$

and so

$$a_m=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

Finally

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \ldots\ldots\ldots\ldots$$

19

$$\boxed{\int_{-\pi}^{\pi} f(x)\sin mx\,\mathrm{d}x = b_m\pi}$$

Because

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\sin mx\,\mathrm{d}x\\
&= \int_{-\pi}^{\pi}\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\right)\sin mx\,\mathrm{d}x\\
&= \frac{a_0}{2}\int_{-\pi}^{\pi}\sin mx\,\mathrm{d}x+\sum_{n=1}^{\infty}a_n\int_{-\pi}^{\pi}\cos nx\sin mx\,\mathrm{d}x\\
&\qquad+\sum_{n=1}^{\infty}b_n\int_{-\pi}^{\pi}\sin nx\sin mx\,\mathrm{d}x\\
&= \frac{a_0}{2}\times 0+\sum_{n=1}^{\infty}a_n\times 0+\sum_{n=1}^{\infty}b_n\pi\delta_{n,m}\\
&= b_m\pi
\end{aligned}$$

and so

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,\mathrm{d}x$$

Summary

20

Given that certain conditions are satisfied, if $f(x)$ is defined on the interval $-\pi \le x \le \pi$ and where $f(x+2n\pi) = f(x)$ then

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

This is the **Fourier series** expansion of $f(x)$ where the a_n and b_n are constants called the *Fourier coefficients* and where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x,\ n = 0,\ 1,\ 2,\ldots$$

Look in particular at the constant function $f(x) = c$ which can be considered as a periodic function with any period we wish to choose. Choosing the period to be 2π then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} c\cos nx\,\mathrm{d}x = \frac{c}{\pi}\int_{-\pi}^{\pi}\cos nx\,\mathrm{d}x = 2c\delta_{n,0}.$$

That is $a_0 = 2c$ so $c = \dfrac{a_0}{2}$ as expected.

Also

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} c\sin nx\,\mathrm{d}x = \frac{c}{\pi}\int_{-\pi}^{\pi}\sin nx\,\mathrm{d}x = 0$$

From this we see that we have two choices to represent the Fourier series. We can either write

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\{a_n\cos nx+b_n\sin nx\}$$

where

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx \text{ and } b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

or we can write

$$f(x)=\sum_{n=0}^{\infty}\{a_n\cos nx+b_n\sin nx\}$$

where

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx,\ a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx\quad(n\neq 0)$$

$$\text{and } b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx.$$

We choose the former and so avoid having a separate integral for a_0.

Dirichlet conditions

21

If a function $f(x)$ is such that

(a) $f(x)$ is defined, single-valued and periodic with period 2π
(b) $f(x)$ and $f'(x)$ have at most a finite number of finite discontinuities over a single period – that is they are *piecewise continuous*

then the series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\{a_n\cos nx+b_n\sin nx\}$$

where $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$ and $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$ converges to $f(x)$ when $(x, f(x))$ is a point of continuity.

The Dirichlet conditions are sufficient for the Fourier series to represent $f(x)$ not only at a point of continuity but, with a slight modification, also at a point of discontinuity, as we shall see later in Frame 36. Also the periodicity of the function need not be restricted to 2π, as we shall see from Frame 38 onwards.

Note that these conditions, while being sufficient, are not necessary because there are functions that do not satisfy these conditions which still possess a convergent Fourier series. However, the cases met in science and engineering do generally meet these conditions.

▶

Exercise

If the following functions are defined over the interval $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$, state whether or not each function can be represented by a Fourier series.

1 $f(x) = x^3$

2 $f(x) = 4x - 5$

3 $f(x) = \dfrac{2}{x}$

4 $f(x) = \dfrac{1}{x-5}$

5 $f(x) = \tan x$

6 $f(x) = y$ where $x^2 + y^2 = 9$

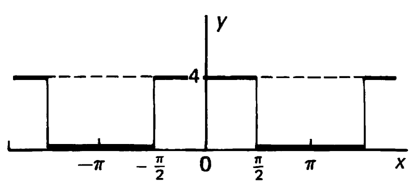
22

1 Yes	**4** Yes
2 Yes	**5** No: infinite discontinuity at $x = \pi/2$
3 No: infinite discontinuity at $x = 0$	**6** No: two valued

On then

23

Example 1



Find the Fourier series for the function shown.

Consider one cycle between $x = -\pi$ and $x = \pi$.

The function can be defined by $f(x) = \begin{cases} 0 & -\pi < x < -\dfrac{\pi}{2} \\ 4 & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} < x < \pi \end{cases}$

$$f(x+2\pi) = f(x).$$

(a) As before $f(x) = \dfrac{1}{2}a_0 + \displaystyle\sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$

The expression for a_0 is ....

24

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x}$$

This gives

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} 0\,\mathrm{d}x + \int_{-\pi/2}^{\pi/2} 4\,\mathrm{d}x + \int_{\pi/2}^{\pi} 0\,\mathrm{d}x\right\}$$

$$= \frac{1}{\pi}\Big[4x\Big]_{-\pi/2}^{\pi/2} \qquad \therefore\ a_0 = 4$$

(b) *To find a_n*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$\therefore\ a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2}(0)\cos nx\,\mathrm{d}x + \int_{-\pi/2}^{\pi/2} 4\cos nx\,\mathrm{d}x + \int_{\pi/2}^{\pi}(0)\cos nx\,\mathrm{d}x\right\}$$

$$\therefore\ a_n = \dots\dots\dots\dots$$

25

$$\boxed{a_n = \frac{8}{\pi n}\sin\frac{n\pi}{2}}$$

Then considering different integer values of n, we have

If n is even $\qquad a_n = 0$

If $n = 1, 5, 9, \dots \qquad a_n = \dfrac{8}{n\pi}$

If $n = 3, 7, 11, \dots \qquad a_n = -\dfrac{8}{n\pi}$

We keep these in mind while we find b_n.

(c) *To find b_n*

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \dots\dots\dots\dots$$

$$\boxed{b_n = 0}$$

Because we have

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2}(0)\sin nx\,\mathrm{d}x + \int_{-\pi/2}^{\pi/2}4\sin nx\,\mathrm{d}x + \int_{\pi/2}^{\pi}(0)\sin nx\,\mathrm{d}x\right\}$$

$$= \frac{4}{\pi}\int_{-\pi/2}^{\pi/2}\sin nx\,\mathrm{d}x = \frac{4}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi/2}^{\pi/2}$$

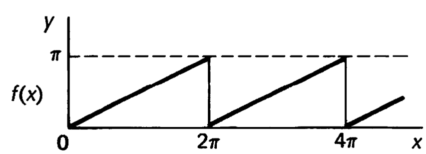
$$= -\frac{4}{n\pi}\left\{\cos\frac{n\pi}{2} - \cos\left(\frac{-n\pi}{2}\right)\right\} = 0 \qquad \therefore\ b_n = 0$$

So with $a_0 = 4$; a_n as stated above; $b_n = 0$; the Fourier series is

$$f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \dots\right\}$$

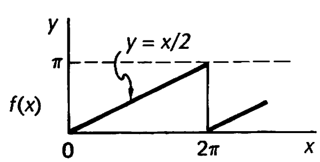
In this particular example, there are, in fact, no sine terms.

Example 2



Determine the Fourier series to represent the periodic function shown.

It is more convenient here to take the limits as 0 to 2π.



The function can be defined as

$$f(x) = \frac{x}{2} \qquad 0 < x < 2\pi$$

$$f(x + 2\pi) = f(x) \quad \text{i.e. period} = 2\pi.$$

Now to find the coefficients.

(a) $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)\mathrm{d}x = \frac{1}{4\pi}\left[x^2\right]_0^{2\pi}$$

$$= \pi \qquad \therefore\ a_0 = \pi$$

(b) $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cos nx\,\mathrm{d}x$$

$= \dots\dots\dots\dots$ (integrating by parts)

27

$$\boxed{a_n = 0}$$

Because

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\cos nx\,dx = \frac{1}{2\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right\}$$

$$= \frac{1}{2\pi}\left\{(0-0) - \frac{1}{n}(0)\right\} = 0 \qquad\qquad \therefore\ a_n = 0$$

(c) $b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\sin nx\,dx = \dots\dots\dots\dots$

28

$$\boxed{b_n = -\frac{1}{n}}$$

Straightforward integration by parts, as for a_n, gives the result stated. So we now have

$$a_0 = \dots\dots\dots\dots;\quad a_n = \dots\dots\dots\dots;\quad b_n = \dots\dots\dots\dots$$

29

$$\boxed{a_0 = \pi;\quad a_n = 0;\quad b_n = -\frac{1}{n}}$$

Now the general expression for a Fourier series is

$$\dots\dots\dots\dots$$

30

$$\boxed{f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n\cos nx + b_n\sin nx\}}$$

Therefore in this case

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\{b_n\sin nx\} \qquad \text{because } a_n = 0$$

$$= \frac{\pi}{2} + \left\{-\frac{1}{1}\sin x - \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x - \dots\right\}$$

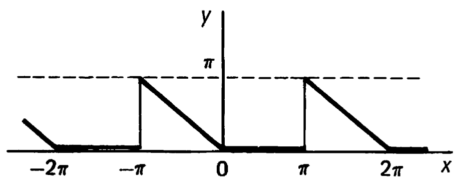
$$\therefore\ f(x) = \frac{\pi}{2} - \left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots\right\}$$

Note that in this example, the series contains a constant term and sine terms only.

▶

Example 3

Find the Fourier series for the function defined by



$$f(x) = -x \qquad -\pi < x < 0$$
$$f(x) = 0 \qquad 0 < x < \pi$$
$$f(x + 2\pi) = f(x).$$

The general expressions for a_0, a_n, b_n are

$a_0 = \ldots\ldots\ldots\ldots$
$a_n = \ldots\ldots\ldots\ldots$
$b_n = \ldots\ldots\ldots\ldots$

31

$$\boxed{\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \end{aligned}}$$

With that reminder, in this example $a_0 = \ldots\ldots\ldots\ldots$

32

$$\boxed{a_0 = \frac{\pi}{2}}$$

Because

(a) $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \dfrac{1}{\pi}\int_{-\pi}^{0} (-x)\,\mathrm{d}x = \dfrac{1}{\pi}\left[-\dfrac{x^2}{2}\right]_{-\pi}^{0} = \dfrac{\pi}{2}$

(b) To find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \ldots\ldots\ldots\ldots$$

33

$$a_n = -\frac{2}{\pi n^2} \quad (n \text{ odd}); \quad 0 \quad (n \text{ even})$$

Because

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\cos nx\,dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx$$

$$= -\frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin nx\,dx\right\}$$

$$= -\frac{1}{\pi}\left\{(0-0) - \frac{1}{n}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0}\right\} = -\frac{1}{\pi n^2}\{1-\cos n\pi\}$$

But $\cos n\pi = 1$ (n even) or -1 (n odd)

$$\therefore\ a_n = -\frac{2}{\pi n^2} \quad (n \text{ odd}) \quad \text{or } 0 \quad (n \text{ even})$$

(c) *Now to find b_n*. Working as for a_n, we obtain

$$b_n = \ldots\ldots\ldots\ldots$$

34

$$b_n = -\frac{1}{n} \quad (n \text{ even}) \quad \text{or} \quad \frac{1}{n} \quad (n \text{ odd})$$

Because

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\sin nx\,dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx$$

$$= -\frac{1}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0}\cos nx\,dx\right\}$$

$$= -\frac{1}{\pi}\left\{\frac{\pi\cos n\pi}{n} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}\right\} = -\frac{\cos n\pi}{n}$$

$$\therefore\ b_n = -\frac{1}{n} \quad (n \text{ even}); \quad \frac{1}{n} \quad (n \text{ odd})$$

So we have $a_0 = \frac{\pi}{2}$; $a_n = 0$ (n even) or $-\frac{2}{\pi n^2}$ (n odd)

$$b_n = -\frac{1}{n} \quad (n \text{ even}) \quad \text{or} \quad \frac{1}{n} \quad (n \text{ odd})$$

$$\therefore\ f(x) = \ldots\ldots\ldots\ldots$$

Complete the series

35

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \dots\right) + \left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\right)$$

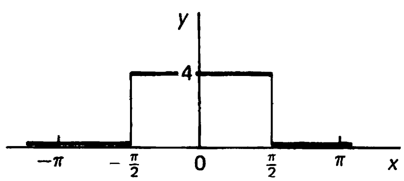
It is just a case of substituting $n = 1, 2, 3$, etc.

In this particular example, we have a constant term and both sine and cosine terms.

Effect of harmonics

It is interesting to see just how accurately the Fourier series represents the function with which it is associated. The complete representation requires an infinite number of terms, but we can, at least, see the effect of including the first few terms of the series.

Let us consider the waveform shown. We established earlier in Frames 23–26 that the function



$$f(x) = \begin{cases} 0 & -\pi < x < -\frac{\pi}{2} \\ 4 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

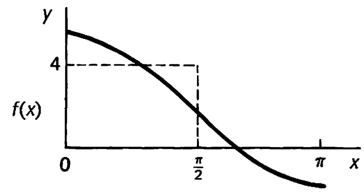
$$f(x + 2\pi) = f(x)$$

gives the Fourier series

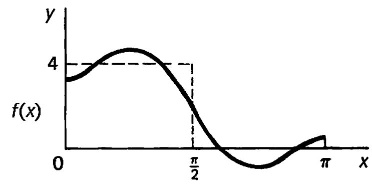
$$f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \dots\right\}$$

If we start with just one cosine term, we can then see the effect of including subsequent harmonics. Let us restrict our attention to just the right-hand half of the symmetrical waveform. Detailed plotting of points gives the development on the next page.

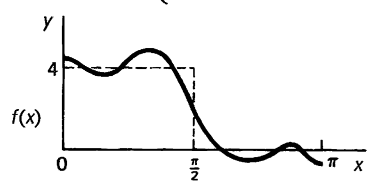
1 $f(x) = 2 + \frac{8}{\pi}\cos x$



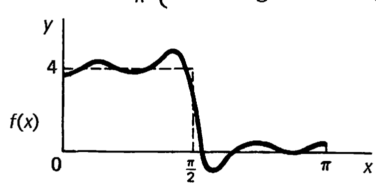
2 $f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x\right\}$



3 $f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x\right\}$



4 $f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x\right\}$



As the number of terms is increased, the graph gradually approaches the shape of the original square waveform. The ripples increase in number and, apart from the one nearest to the step, decrease in amplitude. A perfectly square waveform is unattainable in practice. For practical purposes, the first few terms normally suffice to give an accuracy of acceptable level.

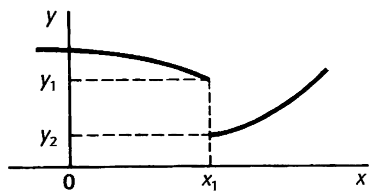
Gibbs' phenomenon

You will notice from the previous two diagrams that near the discontinuity, as more terms are taken into acount, the series tends to overshoot on one side and undershoot on the other. This over and undershooting on either side of the discontinuity does not go away as the number of terms in the Fourier series that are taken into account is increased, rather it tends to two spikes on either side of the discontinuity. This effect is called the *Gibbs' phenomenon.*

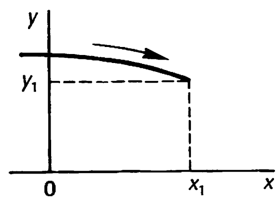
Sum of a Fourier series at a point of discontinuity

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

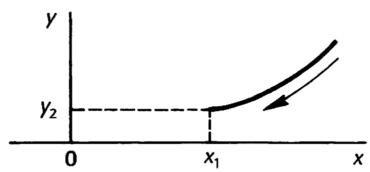
If $f(x)$ is continuous at $x = x_1$, the series converges to the value $f(x_1)$ as the number of terms included increases to infinity. A particular point of interest occurs at a point of finite discontinuity or 'jump' of the function $y = f(x)$.



As $x \to x_1$, the expression $f(x)$ approaches y_1 or y_2 depending on the direction of approach.



If we approach $x = x_1$ from below that value, the limiting value of $f(x)$ is y_1.



If we approach $x = x_1$ from above that value, the limiting value of $f(x)$ is y_2.

To distinguish between these two values we write

$y_1 = f(x_1 - 0)$ denoting immediately before $x = x_1$

$y_2 = f(x_1 + 0)$ denoting immediately after $x = x_1$.

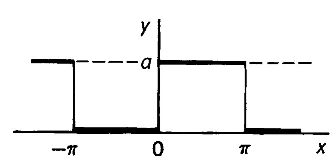
In fact, if we substitute $x = x_1$ in the Fourier series for $f(x)$, it can be shown that the series converges to the value $\frac{1}{2}\{f(x_1 - 0) + f(x_1 + 0)\}$ i.e. $\frac{1}{2}(y_1 + y_2)$, the average of y_1 and y_2.

Example

Consider the function

$$f(x) = \begin{cases} 0 & -\pi < x < \pi \\ a & \pi < x < 2\pi \end{cases}$$

$$f(x + 2\pi) = f(x).$$



First of all, determine the Fourier series to represent the function. There are no snags.

$f(x) = \ldots\ldots\ldots\ldots$

37

$$f(x) = \frac{a}{2} + \frac{2a}{\pi}\{\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \dots\}$$

Check the working

(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} a\,dx = \frac{1}{\pi}\Big[ax\Big]_0^{\pi} = a \qquad \therefore\ a_0 = a$

(b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} a\cos nx\,dx$

$= \frac{a}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = 0 \qquad \therefore\ a_n = 0$

(c) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} a\sin nx\,dx$

$= \frac{a}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{a}{n\pi}(1 - \cos n\pi) = \frac{a}{n\pi}(1 - (-1)^n)$

and because

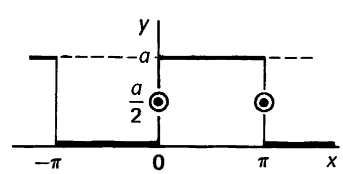
$\cos n\pi = 1 \quad (n \text{ even}) \text{ and } -1 \quad (n \text{ odd})$

$b_n = 0 \quad (n \text{ even}); \quad \frac{2a}{n\pi} \quad (n \text{ odd})$

$\therefore\ f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} b_n \sin nx$

$\therefore\ f(x) = \frac{a}{2} + \frac{2a}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\}$

A finite discontinuity, or 'jump', occurs at $x = 0$. If we substitute $x = 0$ in the series obtained, all the sine terms vanish and we get $f(x) = a/2$, which is, in fact, the average of the two function values at $x = 0$.



Note also that at $x = \pi$, another finite discontinuity occurs and substituting $x = \pi$ in the series gives the same result.

Now on to something new

Functions with periods other than 2π

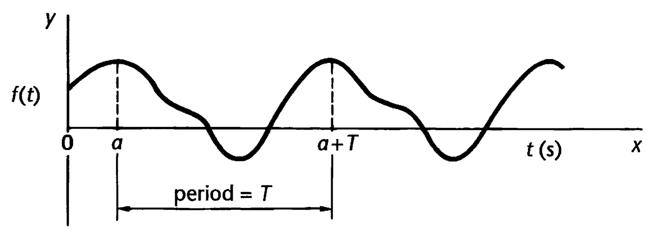
So far, we have considered functions $f(x)$ with period 2π. In practice, we often encounter functions defined over periodic intervals other than 2π, e.g. from 0 to T, $-\frac{T}{2}$ to $\frac{T}{2}$, etc.

Functions with period T

If $y = f(x)$ is defined in the range $-\frac{T}{2}$ to $\frac{T}{2}$, i.e. has a period T, we can convert this to an interval of 2π by changing the units of the independent variable.

In many practical cases involving physical oscillations, the independent variable is time (t) and the periodic interval is normally denoted by T, i.e.

$$f(t+T) = f(t)$$



Each cycle is therefore completed in T seconds and the frequency f **hertz** (oscillations per second) of the periodic function is therefore given by $f = \frac{1}{T}$. If the angular velocity, ω radians per second, is defined by $\omega = 2\pi f$, then

$$\omega = \frac{2\pi}{T} \quad \text{and} \quad T = \frac{2\pi}{\omega}.$$

The angle, x radians, at any time t is therefore $x = \omega t$ and the Fourier series to represent the function can be expressed as

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\} \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{a_n \cos \frac{2n\pi t}{T} + b_n \sin \frac{2n\pi t}{T}\right\} \end{aligned}$$

39 Fourier coefficients

With the new variable, the Fourier coefficients become

$$f(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos n\omega t + b_n \sin n\omega t\}$$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt \qquad = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\cos n\omega t\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin n\omega t\,dt.$$

We can see that there is very little difference between these expressions and those that have gone before. The limits can, of course, be 0 to T, $-\dfrac{T}{2}$ to $\dfrac{T}{2}$, $-\dfrac{\pi}{\omega}$ to $\dfrac{\pi}{\omega}$, 0 to $\dfrac{2\pi}{\omega}$ etc. as is convenient, so long as they cover a complete period.

Example

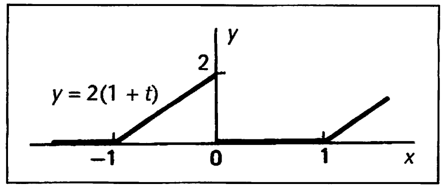
Determine the Fourier series for a periodic function defined by

$$f(t) = \begin{cases} 2(1+t) & -1 < t < 0 \\ 0 & 0 < t < 1 \end{cases}$$

$$f(t+2) = f(t)$$

The first step is to sketch the waveform which is ....

40



We have

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}\right\}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos n\pi t + b_n \sin n\pi t\} \quad \text{because } T = 2$$

Therefore

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt = \int_{-1}^{1} f(t)\,dt = \int_{-1}^{0} 2(1+t)\,dt + \int_{0}^{1} (0)\,dt$$

$$= \left[2t + t^2\right]_{-1}^{0} = 1$$

and

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\pi t\,dt = \int_{-1}^{1} f(t)\cos n\pi t\,dt$$

$$= \int_{-1}^{0} 2(1+t)\ \cos n\pi t\,dt = \dots\dots\dots\dots$$

41

$$\boxed{a_n = 0\ (n \text{ even});\quad a_n = \frac{4}{n^2\pi^2}\ (n \text{ odd})}$$

Because

$$a_n = \int_{-1}^{0} 2(1+t)\cos n\pi t\,dt$$

$$= 2\left\{\left[(1+t)\frac{\sin\ n\pi t}{n\pi}\right]_{-1}^{0} - \frac{1}{n\pi}\int_{-1}^{0} \sin\ n\pi t\,dt\right\}$$

$$= 2\left\{(0-0) - \frac{1}{n\pi}\left[-\frac{\cos n\pi t}{n\pi}\right]_{-1}^{0}\right\} = \frac{2}{n^2\pi^2}(1 - \cos n\pi)$$

$$= \frac{2}{n^2\pi^2}(1 - (-1)^n)$$

so that

$$a_n = 0 \quad (n \text{ even}), \quad a_n = \frac{4}{n^2\pi^2} \quad (n \text{ odd})$$

Now for b_n

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2n\pi t}{T}\,dt = \dots\dots\dots\dots$$

42

$$\boxed{b_n = -\frac{2}{n\pi}}$$

Because

$$\begin{aligned} b_n &= \int_{-1}^{0} 2(1+t)\sin n\pi t\,\mathrm{d}t \\ &= 2\left\{\left[(1+t)\frac{-\cos n\pi t}{n\pi}\right]_{-1}^{0} + \frac{1}{n\pi}\int_{-1}^{0}\cos n\pi t\,\mathrm{d}t\right\} \\ &= 2\left\{-\frac{1}{n\pi} + \left[\frac{\sin n\pi t}{n\pi}\right]_{-1}^{0}\right\} = -\frac{2}{n\pi} + \frac{2}{n^2\pi^2}(\sin n\pi) = -\frac{2}{n\pi} \end{aligned}$$

So the first few terms of the series give

$$f(t) = \ldots\ldots\ldots\ldots$$

43

$$\boxed{\begin{aligned} f(t) = \frac{1}{2} + \frac{4}{\pi^2}&\left\{\cos\pi t + \frac{1}{9}\cos 3\pi t + \frac{1}{25}\cos 5\pi t + \ldots\right\} \\ &-\frac{2}{\pi}\left\{\sin\pi t + \frac{1}{2}\sin 2\pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{4}\sin 4\pi t + \ldots\right\} \end{aligned}}$$

The Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n\cos n\omega t + b_n\sin n\omega t\}$$

can also be written in the form

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} B_n\sin(n\omega t + \phi_n)$$

Comparing these two expressions we see that $A_0 = a_0$, $B_n\sin\phi_n = a_n$ and $B_n\cos\phi_n = b_n$. From this it follows that

$$B_n = \ldots\ldots\ldots\ldots \text{ and } \phi_n = \ldots\ldots\ldots\ldots$$

44

$$\boxed{B_n = \sqrt{a_n^2 + b_n^2};\quad \phi_n = \arctan\left(\frac{a_n}{b_n}\right)}$$

So

$B_1\sin(\omega t + \phi_1)$	is the first harmonic or fundamental (lowest frequency)
$B_2\sin(2\omega t + \phi_2)$	is the second harmonic (frequency twice that of the fundamental)
$B_n\sin(n\omega t + \phi_n)$	is the nth harmonic (frequency n times that of the fundamental).

And for the series to converge, the values of B_n must eventually decrease with higher-order harmonics, i.e. $B_n \to 0$ as $n \to \infty$.

Odd and even functions

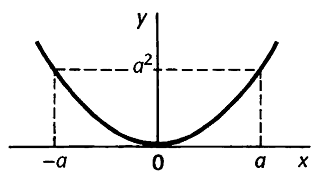
45

(a) *Even functions*

A function $f(x)$ is said to be *even* if

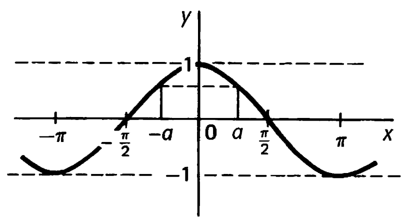
$$f(-x) = f(x)$$

i.e. the function value for a particular negative value of x is the same as that for the corresponding positive value of x. The graph of an even function is therefore *symmetrical about the y-axis.*



$y = f(x) = x^2$ is an even function because

$$f(-2) = 4 = f(2)$$
$$f(-3) = 9 = f(3) \quad \text{etc.}$$



$y = f(x) = \cos x$ is an even function because

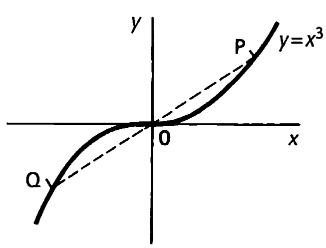
$$\cos(-x) = \cos x$$
$$f(-a) = \cos a = f(a).$$

(b) *Odd functions*

A function $f(x)$ is said to be *odd* if

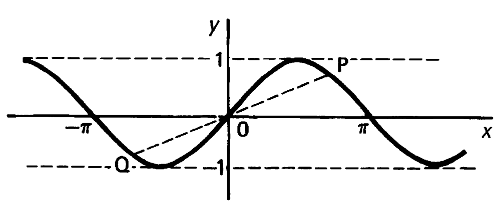
$$f(-x) = -f(x)$$

i.e. the function value for a particular negative value of x is numerically equal to that for the corresponding positive value of x but opposite in sign. The graph of an odd function is thus *symmetrical about the origin.*



$y = f(x) = x^3$ is an odd function because

$$f(-2) = -8 = -f(2)$$
$$f(-5) = -125 = -f(5) \quad \text{etc.}$$

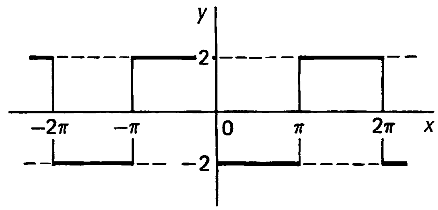


$y = f(x) = \sin x$ is an odd function because

$$\sin(-x) = -\sin x$$
$$f(-a) = -f(a).$$

So, for an even function $f(-x) = f(x)$, symmetrical about the y-axis
for an odd function $f(-x) = -f(x)$, symmetrical about the origin.

Example 1

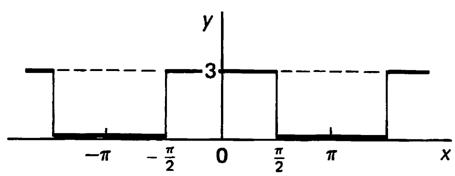


$f(x)$ shown by the waveform is therefore an .... function because it is ....

46

odd; symmetrical about the origin, i.e. $f(-x) = -f(x)$

Example 2

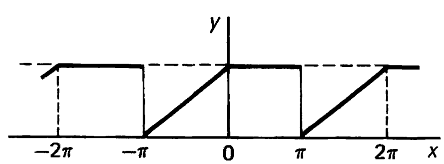


Hence the waveform of $y = f(x)$ depicts an .... function, because it is ....

47

even; symmetrical about the y-axis, i.e. $f(-x) = f(x)$

Example 3



In this case, the waveform shows a function that is .... because ....

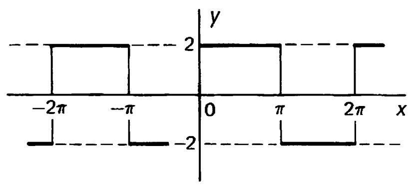
48

neither even nor odd; not symmetrical about either the y-axis or the origin

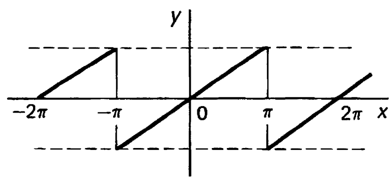
Exercise

State whether each of the following functions is odd, even, or neither.

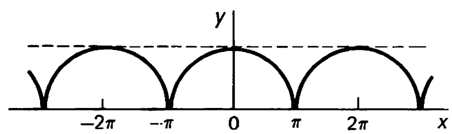
1



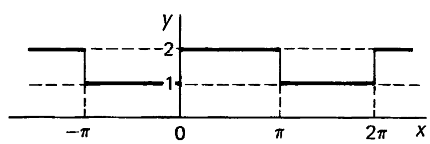
2



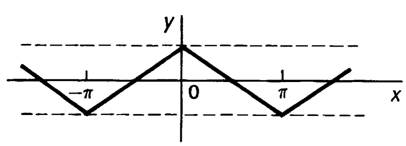
3



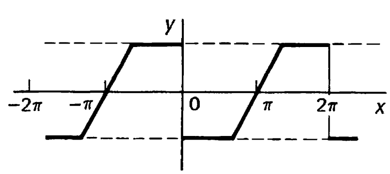
4



5



6



49

1 odd	**2** odd	**3** even
4 neither	**5** even	**6** odd

We shall shortly see that a knowledge of odd and even functions can save a lot of unnecessary calculation.

First, however, let us consider products of odd and even functions in the next frame

50

Products of odd and even functions

The rules closely resemble the elementary rules of signs.

$$\begin{aligned} &(\text{even}) \times (\text{even}) = (\text{even}) \quad \text{like} \quad &(+) \times (+) = (+) \\ &(\text{odd}) \times (\text{odd}) = (\text{even}) &(-) \times (-) = (+) \\ &(\text{odd}) \times (\text{even}) = (\text{odd}) &(-) \times (+) = (-). \end{aligned}$$

The results can easily be proved.

(a) *Two even functions*

Let $F(x) = f(x)g(x)$ where $f(x)$ and $g(x)$ are even functions.

Then $F(-x) = f(-x)g(-x) = f(x)g(x)$ since $f(x)$ and $g(x)$ are even

$\therefore\ F(-x) = F(x)$ i.e. $F(x)$ is even

(b) *Two odd functions*

Let $F(x) = f(x)g(x)$ where $f(x)$ and $g(x)$ are odd functions.

Then $F(-x) = f(-x)g(-x)$

$= \{-f(x)\}\{-g(x)\}$ since $f(x)$ and $g(x)$ are odd

$= f(x)g(x) = F(x)$

$\therefore\ F(-x) = F(x)$ i.e. $F(x)$ is even

Finally

(c) *One odd and one even function*

Let $F(x) = f(x)g(x)$ where $f(x)$ is odd and $g(x)$ even.

Then $F(-x) = f(-x)g(-x) = -f(x)g(x) = -F(x)$

$\therefore\ F(-x) = -F(x)$ i.e. $F(x)$ is odd

So if $f(x)$ and $g(x)$ are both even, then $f(x)g(x)$ is even
and if $f(x)$ and $g(x)$ are both odd, then $f(x)g(x)$ is even
but if either $f(x)$ or $g(x)$ is even and the other odd, then $f(x)g(x)$ is odd.

Now for a short exercise, so move on

51

Exercise

State whether each of the following products is odd, even, or neither.

1 $x^2 \sin 2x$

2 $x^3 \cos x$

3 $\cos 2x \cos 3x$

4 $x \sin nx$

5 $3 \sin x \cos 4x$

6 $(2x + 3) \sin 4x$

7 $\sin^2 x \cos 3x$

8 $x^3 e^x$

9 $(x^4 + 4) \sin 2x$

10 $\dfrac{1}{x + 2} \cosh x$

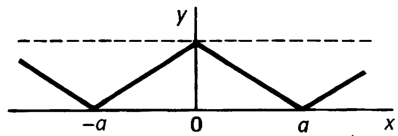
Finish all ten and then check with the next frame

52

1	odd (E)(O) = (O)	**6**	neither (N)(O) = (N)
2	odd (O)(E) = (O)	**7**	even (E)(E) = (E)
3	even (E)(E) = (E)	**8**	neither (O)(N) = (N)
4	even (O)(O) = (E)	**9**	odd (E)(O) = (O)
5	odd (O)(E) = (O)	**10**	neither (N)(E) = (N)

Two useful facts emerge from odd and even functions. Thinking in terms of areas under the graphs

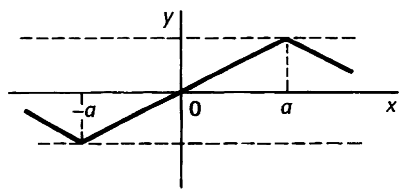
(a)



For an *even* function

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(b)



For an *odd* function

$$\int_{-a}^{a} f(x)dx = 0$$

We can now look at two important theorems concerning odd and even functions.

Theorem 1

If $f(x)$ is defined over the interval $-\pi < x < \pi$ and $f(x)$ is *even*, then the Fourier series for $f(x)$ contains *cosine terms* only. Included in this is a_0 which may be regarded as $a_n \cos nx$ with $n = 0$.

Proof: Since $f(x)$ is even, $\displaystyle\int_{-\pi}^{0} f(x)\,dx = \int_{0}^{\pi} f(x)\,dx$

(a) $\displaystyle a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx \qquad \therefore\ a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$

(b) $\displaystyle a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$

But $f(x)\cos nx$ is the product of two even functions and therefore itself even.

$$\therefore\ a_n = \ldots\ldots\ldots\ldots$$

53

$$\boxed{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x}$$

Because as the integrand is even,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x.$$

(c) $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$

Arguing along similar lines, this gives $b_n = \ldots\ldots\ldots\ldots$

54

$$\boxed{b_n = 0}$$

Because, since $f(x)\sin nx$ is the product of an even function and an odd function, it is itself odd.

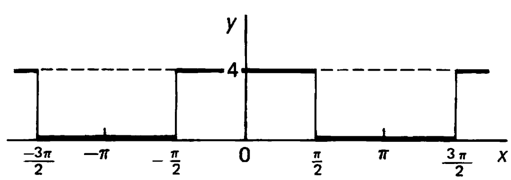
$$\therefore\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = 0.\quad \therefore\ b_n = 0$$

Therefore, there are no sine terms in the Fourier series for $f(x)$.

Now for an example.

Example

The waveform shown is symmetrical about the y-axis. The function is therefore even and there will be no sine terms in the series.



$$\therefore\ f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

(a) $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \dfrac{2}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \dfrac{2}{\pi}\int_0^{\pi/2} 4\,\mathrm{d}x = \dfrac{2}{\pi}\Big[4x\Big]_0^{\pi/2} = 4$

(b) $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \dfrac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$

$= \ldots\ldots\ldots\ldots$ Finish the integration.

55

$$a_n = 0 \quad (n \text{ even}); \quad a_n = \frac{8}{\pi n} \quad (n = 1, 5, 9, \ldots);$$
$$a_n = -\frac{8}{\pi n} \quad (n = 3, 7, 11, \ldots)$$

Because

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} 4\cos nx\,dx$$
$$= \frac{8}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi/2} = \frac{8}{\pi n}\sin\frac{n\pi}{2}$$

But $\sin\frac{n\pi}{2} = 0$ for n even

$= 1$ for $n = 1, 5, 9, \ldots$

$= -1$ for $n = 3, 7, 11, \ldots$ Hence the result stated.

(c) We know that $b_n = 0$, because $f(x)$ is an even function. Therefore, the required series is

$$f(x) = \ldots\ldots\ldots\ldots$$

56

$$f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \ldots\right\}$$

If you care to look back to Example 1 in Frame 23, you will see how much time and effort we have saved by not having to evaluate b_n.

A similar theorem applies to odd functions.

Theorem 2

If $f(x)$ is an *odd* function defined over the interval $-\pi < x < \pi$, then the Fourier series for $f(x)$ contains *sine terms* only.

Proof: Since $f(x)$ is an odd function, $\int_{-\pi}^{0} f(x)\,dx = -\int_0^{\pi} f(x)\,dx$.

(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$. But $f(x)$ is odd $\therefore\ a_0 = 0$

(b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

Remembering that $f(x)$ is odd and $\cos nx$ is even, the product $f(x)\cos nx$ is $\ldots\ldots\ldots\ldots$

57

$$\boxed{\text{odd}}$$

$$\therefore\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\text{odd function})\,dx = 0$$

$\therefore\ a_n = 0$ (including $a_0 = 0$)

Now for b_n we have

(c) $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ and because $f(x)$ and $\sin nx$ are each odd, the product $f(x)\sin nx$ is ....

58

$$\boxed{\text{even}}$$

$$\text{Then } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\text{even function})\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

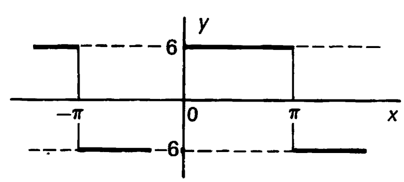
$$\therefore\ b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

So, if $f(x)$ is odd, $a_0 = 0$; $a_n = 0$; $b_n = \dfrac{2}{\pi}\displaystyle\int_{0}^{\pi} f(x)\sin nx\,dx$

i.e. the Fourier series contains sine terms only.

Example

Consider the function shown.



$$f(x) = -6 \qquad -\pi < x < 0$$
$$f(x) = 6 \qquad 0 < x < \pi$$
$$f(x + 2\pi) = f(x).$$

Before we do any evaluation, we can see that this is .... and therefore ....

59

$$\boxed{\text{an odd function;}\quad \text{sine terms only, i.e. } a_0 = 0 \text{ and } a_n = 0}$$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$ $f(x)\sin nx$ is a product of two odd functions and is therefore even.

$$\therefore\ b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx = \text{...........}$$

60

$$\boxed{b_n = 0 \quad (n \text{ even}) \quad \text{or} \quad \frac{24}{\pi n} \quad (n \text{ odd})}$$

Because

$$b_n = \frac{2}{\pi}\int_0^{\pi} 6 \sin nx \, dx = \frac{12}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{12}{\pi n}(1 - \cos n\pi).$$

Hence the result stated above.

So the series is $f(x) = \ldots\ldots\ldots\ldots$

61

$$\boxed{f(x) = \frac{24}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right\}}$$

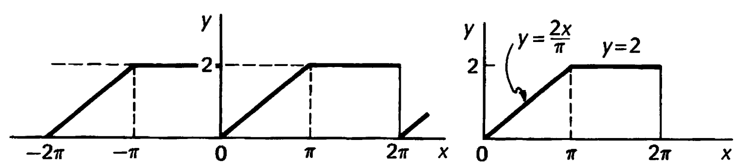
Because $\cos n\pi = (-1)^n$.

Of course, if $f(x)$ is neither an odd nor an even function, then we must obtain expressions for a_0, a_n and b_n in full.

One more example

Example

Determine the Fourier series for the function shown.



This is neither odd nor even. Therefore we must find a_0, a_n and b_n.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

(a) $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\,dx = \dfrac{1}{\pi}\left\{\int_0^{\pi}\frac{2}{\pi}x\,dx + \int_{\pi}^{2\pi} 2\,dx\right\}$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{\pi}\right]_0^{\pi} + \left[2x\right]_{\pi}^{2\pi}\right\} = \frac{1}{\pi}\{\pi + 4\pi - 2\pi\} = 3 \qquad \therefore\ a_0 = 3$$

(b) $a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\cos nx\,dx = \dfrac{1}{\pi}\left\{\int_0^{\pi}\left(\frac{2}{\pi}x\right)\cos nx\,dx + \int_{\pi}^{2\pi} 2\cos nx\,dx\right\}$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi}\left[\frac{x \sin nx}{n}\right]_0^{\pi} - \frac{1}{\pi n}\int_0^{\pi}\sin nx\,dx + \int_{\pi}^{2\pi}\cos nx\,dx\right\}$$

$= \ldots\ldots\ldots\ldots$

Finish it off

62

$$\boxed{a_n = 0 \quad (n \text{ even}); \quad a_n = \frac{-4}{\pi^2 n^2} \quad (n \text{ odd})}$$

Because

$$\begin{aligned}
a_n &= \frac{2}{\pi}\left\{\frac{1}{\pi}(0-0) - \frac{1}{\pi n}\left[-\frac{\cos nx}{n}\right]_0^{\pi} + \left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}\right\} \\
&= \frac{2}{\pi}\left\{-\frac{1}{\pi n^2}(-(-1)^n + 1) + (0-0)\right\} \\
&= -\frac{2}{\pi^2 n^2}(1-(-1)^n)
\end{aligned}$$

and so

$$a_n = 0 \quad (n \text{ even}) \text{ and } a_n = -\frac{4}{\pi^2 n^2} \quad (n \text{ odd})$$

(c) To find b_n, we proceed in the same general manner

$$b_n = \dots\dots\dots\dots$$

Complete it on your own

63

$$\boxed{b_n = -\frac{2}{\pi n}}$$

Here is the working.

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_0^{\pi}\left(\frac{2}{\pi}x\right)\sin nx\,dx + \int_{\pi}^{2\pi} 2\,\sin nx\,dx\right\} \\
&= \frac{2}{\pi}\left\{\frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{1}{\pi n}\int_0^{\pi}\cos nx\,dx + \int_{\pi}^{2\pi}\sin nx\,dx\right\} \\
&= \frac{2}{\pi}\left\{\frac{1}{\pi n}(-\pi\cos n\pi) + \frac{1}{\pi n}\left[\frac{\sin nx}{n}\right]_0^{\pi} + \left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}\right\} \\
&= \frac{2}{\pi}\left\{-\frac{1}{n}\cos n\pi + (0-0) - \frac{1}{n}(\cos\,2\pi n - \cos n\pi)\right\} \\
&= \frac{2}{\pi}\left\{-\frac{1}{n}\cos\,2n\pi\right\} = -\frac{2}{\pi n}\cos\,2n\pi
\end{aligned}$$

But $\cos\,2n\pi = 1$. $\quad \therefore\ b_n = -\dfrac{2}{\pi n}$

So the required series is $f(x) = \dots\dots\dots\dots$

64

$$f(x) = \frac{3}{2} - \frac{4}{\pi^2}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \dots\right\}$$
$$- \frac{2}{\pi}\left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x \dots\right\}$$

At this stage, let us take stock of our findings so far.
If a function $f(x)$ is defined over the range $-\pi$ to π, or any other periodic interval of 2π, then the Fourier series for $f(x)$ is of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

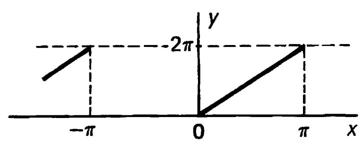
We also know that

(a) if $f(x)$ is an *even* function, the series will contain *no sine terms*
(b) if $f(x)$ is an *odd* function, the series will contain *only sine terms*
(c) if $f(x)$ is *neither odd nor even*, the series will, in general, contain a constant term, cosine terms and sine terms.

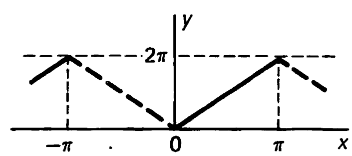
65 Half-range series

Sometimes a function of period 2π is defined over the range 0 to π, instead of the normal $-\pi$ to π, or 0 to 2π. We then have a choice of how to proceed.

For example, if we are told that between $x = 0$ and $x = \pi$, $f(x) = 2x$, then, since the period is 2π, we have no evidence of how the function behaves between $x = -\pi$ and $x = 0$.

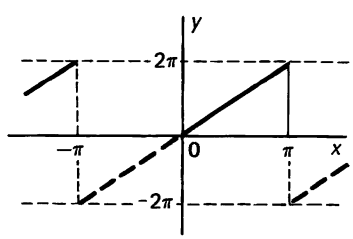


(a)



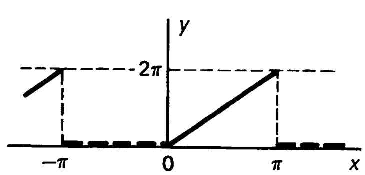
If the waveform were as shown in (a), the function would be an even function, symmetrical about the y-axis and the series would have *only cosine terms* (including possibly a_0).

(b)



On the other hand, if the waveform were as shown in (b), the function would be odd, being symmetrical about the origin and the series would have *only sine terms.*

(c)



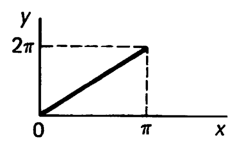
Of course, if we choose something quite different for the waveform between $x = -\pi$ and $x = 0$, then $f(x)$ will be neither odd nor even and the series will then contain ....

66

both sine and cosine terms (including a_0)

In each case, we are making an assumption on how the function behaves between $x = -\pi$ and $x = 0$, and the resulting Fourier series will therefore apply only to $f(x)$ between $x = 0$ and $x = \pi$ for which it is defined. For this reason, such series are called *half-range series.*

Example 1



A function $f(x)$ is defined by

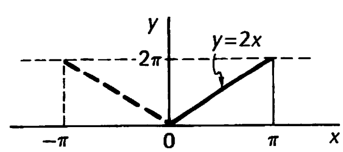
$$f(x) = 2x \qquad 0 < x < \pi$$
$$f(x + 2\pi) = f(x).$$

Obtain a half-range cosine series to represent the function.

To obtain a cosine series, i.e. a series with no sine terms, we need an .... function.

67

even



Therefore, we assume the waveform between $x = -\pi$ and $x = 0$ to be as shown, making the total graph symmetrical about the y-axis.

Now we can find expressions for the Fourier coefficients as usual.

$$a_0 = \ldots\ldots\ldots\ldots$$

68

$a_0 = 2\pi$

Because

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} 2x\,dx = \frac{2}{\pi}\left[x^2\right]_0^{\pi} = 2\pi \qquad \therefore\ a_0 = 2\pi$$

Then we need a_n which is ....

69

$$a_n = 0 \quad (n \text{ even}) \qquad = -\frac{8}{\pi n^2} \quad (n \text{ odd})$$

Because

$$a_n = \frac{2}{\pi}\int_0^{\pi} 2x\cos nx\,dx = \frac{4}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{4}{\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right\}$$

$$= \frac{4}{\pi}\left\{(0-0) - \frac{1}{n}\left[\frac{-\cos nx}{n}\right]_0^{\pi}\right\} = \frac{4}{\pi n^2}(\cos n\pi - 1)$$

$$\cos n\pi = 1 \quad (n \text{ even}) \qquad = -1 \quad (n \text{ odd})$$

$$\therefore\ a_n = 0 \quad (n \text{ even}) \quad \text{and} \quad a_n = -\frac{8}{\pi n^2} \quad (n \text{ odd})$$

All that now remains is b_n which is ....

70

zero, since $f(x)$ is an even function, i.e. $b_n = 0$

So $a_0 = 2\pi$, $a_n = 0$ $(n$ even$)$ or $-\dfrac{8}{\pi n^2}$ $(n$ odd$)$, $b_n = 0$.

Therefore $f(x) =$ ....

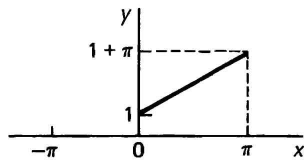
71

$$f(x) = \pi - \frac{8}{\pi}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \dots\right\}$$

Let us look at a further example, so move on to the next frame

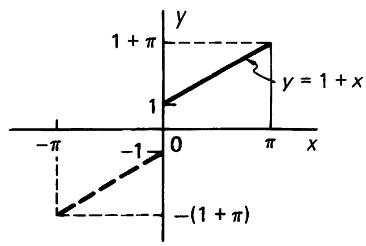
72

Example 2



Determine a half-range sine series to represent the function $f(x)$ defined by

$$f(x) = 1 + x \qquad 0 < x < \pi$$
$$f(x + 2\pi) = f(x).$$



We choose the waveform between $x = -\pi$ and $x = 0$ so that the graph is symmetrical about the origin. The function is then an odd function and the series will contain only sine terms.

$$\therefore\ a_0 = 0 \quad \text{and} \quad a_n = 0$$

b_n can now easily be determined and the required series obtained.

$$f(x) = \ldots\ldots\ldots\ldots$$

73

$$\boxed{\begin{aligned} f(x) = \left(\frac{4}{\pi} + 2\right)&\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right\} \\ &-2\left\{\frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x + \frac{1}{6}\sin 6x + \ldots\right\}\end{aligned}}$$

Check the working.

$$b_n = \frac{2}{\pi}\int_0^{\pi}(1 + x)\sin nx\,\mathrm{d}x = \frac{2}{\pi}\left\{\left[(1 + x)\frac{-\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,\mathrm{d}x\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{1 + \pi}{n}\cos n\pi + \frac{1}{n} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n} - \frac{1 + \pi}{n}\cos n\pi\right\} = \frac{2}{\pi n}\{1 - (1 + \pi)\cos n\pi\}$$

$$\cos n\pi = 1 \quad (n \text{ even}) \qquad = -1 \quad (n \text{ odd})$$

$$\therefore\ b_n = -\frac{2}{n} \quad (n \text{ even}) \qquad = \frac{4 + 2\pi}{\pi n} \quad (n \text{ odd})$$

Substituting in the general expression $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ we have

$$f(x) = \frac{4 + 2\pi}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right\}$$
$$-\,2\left\{\frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x + \frac{1}{6}\sin 6x + \ldots\right\}$$

So a knowledge of odd and even functions and of half-range series saves a deal of unnecessary work on occasions.

Now let us consider the presence of odd or even harmonics, so move on

74 Series containing only odd harmonics or only even harmonics

$$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

If we replace x by $(x + \pi)$, this becomes

$$f(x + \pi) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos n(x + \pi) + b_n \sin n(x + \pi)\}$$

Now $\cos(nx + n\pi) = \cos nx \cos n\pi - \sin nx \sin n\pi$.

But for $n = 1, 2, 3, \dots \quad \sin n\pi = 0$

$\therefore \quad \cos n(x + \pi) = \cos nx \cos n\pi$

Also for $n = 1, 2, 3, \dots \quad \cos n\pi = 1 \quad (n \text{ even}) \quad = -1 \; (n \text{ odd})$.

$$\therefore \quad \cos n(x + \pi) = \cos nx \quad (n \text{ even}) \quad = -\cos nx \quad (n \text{ odd}) \qquad (1)$$

Similarly, $\sin(nx + n\pi) = \sin nx \cos n\pi + \cos nx \sin n\pi$.

Therefore, as before

$$\sin n(x + \pi) = \sin nx \quad (n \text{ even}) \quad = -\sin nx \quad (n \text{ odd}) \qquad (2)$$

$$\therefore \quad f(x + \pi) = \tfrac{1}{2}a_0 - a_1 \cos x + a_2 \cos 2x - a_3 \cos 3x + \dots$$
$$- b_1 \sin x + b_2 \sin 2x - b_3 \sin 3x + \dots$$

But $\quad f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

If $f(x) = f(x + \pi)$, these two series are equal and the odd harmonics that you see differ in sign must be zero.

$$\therefore \quad f(x) = f(x + \pi) = \tfrac{1}{2}a_0 + a_2 \cos 2x + a_4 \cos 4x + \dots$$
$$+ b_2 \sin 2x + b_4 \sin 4x + \dots$$

$\therefore$ If $f(x) = f(x + \pi)$, the Fourier series for $f(x)$ contains even harmonics only.

Similarly, from the same two series above

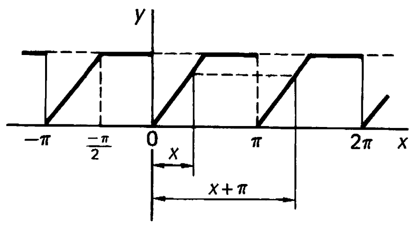
if $f(x) = -f(x + \pi)$, the Fourier series for $f(x)$ contains odd harmonics only.

$$\therefore \quad f(x) = a_1 \cos x + a_3 \cos 3x + \dots + b_1 \sin x + b_3 \sin 3x + \dots$$

Make a note of these two results: you will find them useful

75

Example 1



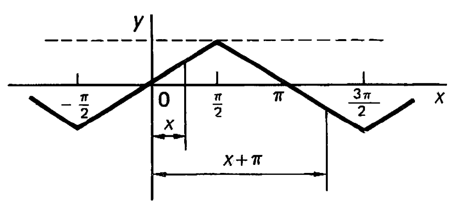
Here $f(x) = f(x + \pi)$

Therefore, the series contains ....

76

$\boxed{\text{even harmonics only}}$

Example 2



Here we see that $f(x) = -f(x + \pi)$.

Therefore, the series contains ....

77

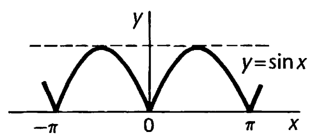
odd harmonics only

Now we can apply our knowledge to date to the following exercise.

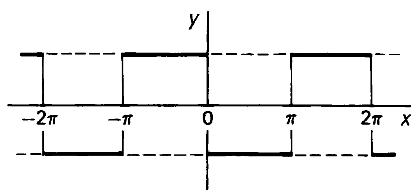
Exercise

From each of the following waveforms, we can describe the nature of the terms in the relevant Fourier series.

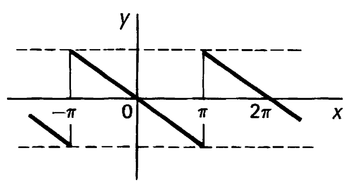
1



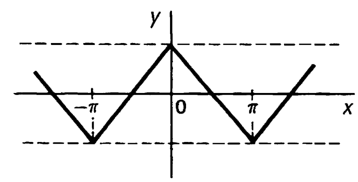
2



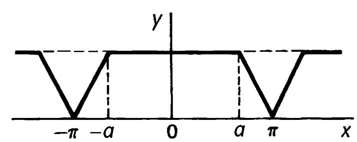
3



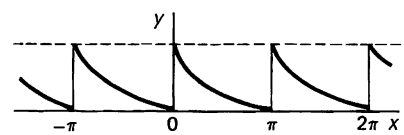
4



5



6



78

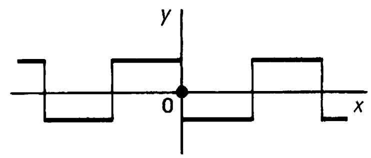
1 cosine terms ($+a_0$) only; even harmonics only
2 sine terms only; odd harmonics only
3 sine terms only; all harmonics
4 cosine terms ($+a_0$) only; odd harmonics only
5 cosine terms ($+a_0$) only; all harmonics
6 a_0, sine and cosine terms; even harmonics only.

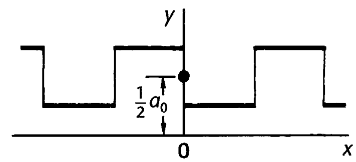
On we go

Significance of the constant term $\frac{1}{2}a_0$

79

We might, at this point, note that the effect of the constant term $\frac{1}{2}a_0$ is to raise, or lower, the whole waveform on the y-axis.

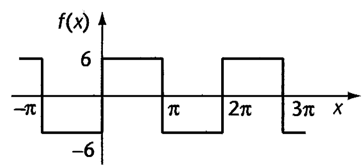




In electrical applications to alternating currents, the constant term $\frac{1}{2}a_0$ of the Fourier series indicates the d.c. component.

For example, from Frames 58–61 we found that the odd square wave

$$f(x) = \begin{cases} -6 & -\pi < x < 0 \\ 6 & 0 < x < \pi \end{cases} \qquad f(x + 2\pi) = f(x)$$

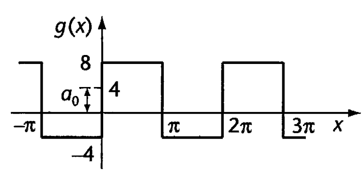


has the Fourier series expansion

$$f(x) = \frac{24}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\}$$

The function $g(x) = 2 + f(x)$ has the Fourier series expansion

$$g(x) = 2 + \frac{24}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\}$$



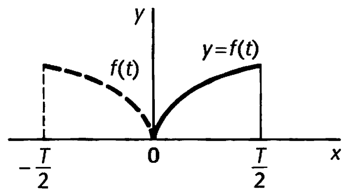
Here $a_0 = 4$.

80

Half-range series with arbitrary period

We now extend the work on half-range sine and cosine series to functions with arbitrary period.

(a) *Even function* Half-range cosine series



$y = f(t) \quad 0 < t < \frac{T}{2}$

$f(t+T) = f(t)$

symmetrical about the y-axis.

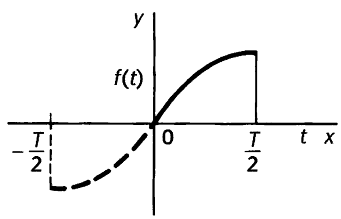
With an even function, we know that $b_n = 0$

$$\therefore\ f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t$$

where $$a_0 = \frac{4}{T}\int_0^{T/2} f(t)\,\mathrm{d}t$$

and $$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\,\mathrm{d}t$$

(b) *Odd function* Half-range sine series



$y = f(t) \quad 0 < t < \frac{T}{2}$

$f(t+T) = f(t)$

symmetrical about the origin.

$\therefore\ a_0 = 0$ and $a_n = 0$

Then $f(t) = \ldots\ldots\ldots\ldots$

and $b_n = \ldots\ldots\ldots\ldots$

81

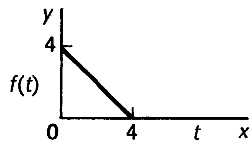
$$\boxed{f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t; \quad b_n = \frac{4}{T}\int_0^{T/2} f(t)\,\sin n\omega t\,\mathrm{d}t}$$

Now for an example or two.

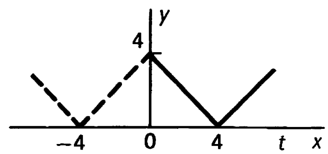
So move on

Example 1 82

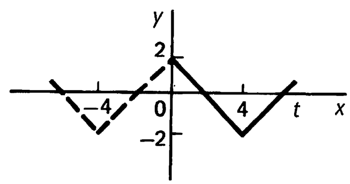
A function $f(t)$ is defined by $f(t) = 4 - t, \quad 0 < t < 4$.



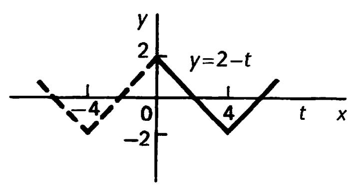
We have to form a half-range cosine series to represent the function in this interval.



First we form an even function, i.e. symmetrical about the y-axis.



Now for a useful little trick. If we lower the waveform 2 units, i.e. to its 'average' position, balanced above and below the x-axis, then in this new position $\frac{1}{2}a_0 = 0$ and we have been saved one set of calculations.



The function is now $y = f_1(t) = 2 - t$ and, for the moment $\frac{1}{2}a_0 = 0$. Also, being an even function $b_n = 0$. All we need to do is to evaluate a_n.

$$\text{So } a_n = \frac{4}{T}\int_0^{T/2} f_1(t)\cos n\omega t\,dt = \frac{4}{8}\int_0^4 (2-t)\cos n\omega t\,dt$$
$$= \ldots\ldots\ldots\ldots$$

83

$$\boxed{a_n = 0 \quad (n \text{ even}) \quad = \frac{1}{n^2\omega^2} \quad (n \text{ odd})}$$

Simple integration by parts gives

$$a_n = \frac{1}{2}\left\{-\frac{2\sin 4n\omega}{n\omega} - \frac{1}{n^2\omega^2}(\cos 4n\omega - 1)\right\}$$

$$\text{But } \omega = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$a_n = \frac{1}{2}\left\{-\frac{2\sin n\pi}{n\omega} - \frac{1}{n^2\omega^2}(\cos n\pi - 1)\right\} \qquad n = 1, 2, 3, \ldots$$

$$\sin n\pi = 0; \quad \cos n\pi = 1 \quad (n \text{ even}); \quad \cos n\pi = -1 \quad (n \text{ odd})$$

$$\therefore\ a_n = 0 \quad (n \text{ even}) \quad \text{and} \quad a_n = \frac{1}{n^2\omega^2} \quad (n \text{ odd})$$

$$\therefore\ f_1(t) = \ldots\ldots\ldots\ldots$$

84

$$\boxed{f_1(t) = \frac{1}{\omega^2}\left\{\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + \dots\right\}}$$

Now if we finally lift the waveform back to its original position by restoring the 2 units (i.e. $\frac{1}{2}a_0 = 2$), the original function is regained with $f(t) = f_1(t) + 2$.

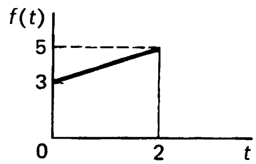
$$\therefore\ f(t) = 2 + \frac{1}{\omega^2}\left\{\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + \dots\right\}$$

where $\omega = \dfrac{\pi}{4}$.

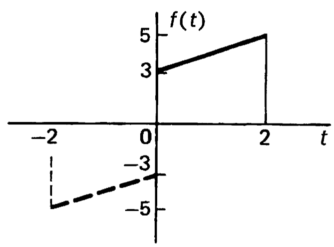
Example 2

A function $f(t)$ is defined by $f(t) = 3 + t \qquad 0 < t < 2$

$$f(t+4) = f(t).$$



Obtain the half-range sine series for the function in this range.



Sine series required. Therefore, we form an odd function, symmetrical about the origin

$a_0 = 0;\ a_n = 0;\ T = 4$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$\therefore\ b_n = \frac{4}{T}\int_0^2 f(t)\ \sin n\omega t\,dt = \int_0^2 (3+t)\sin n\omega t\,dt$$

This you can easily evaluate and then, putting $n = 1, 2, 3, \dots$ obtain the series $f(t) = \dots\dots\dots\dots$

85

$$\boxed{f(t)=\frac{2}{\omega}\left\{4\sin\omega t-\frac{1}{2}\sin 2\omega t+\frac{4}{3}\sin 3\omega t-\frac{1}{4}\sin 4\omega t\ldots\right\}}$$

Straightforward integration by parts gives

$$b_n=\frac{1}{n\omega}(3-5\cos 2n\omega)+\frac{1}{n^2\omega^2}(\sin 2n\omega)$$

But $T=\dfrac{2\pi}{\omega}\quad\therefore\ \omega=\dfrac{2\pi}{T}=\dfrac{\pi}{2}$

$$\therefore\ b_n=\frac{1}{n\omega}(3-5\cos n\pi)+\frac{1}{n^2\omega^2}\sin n\pi$$

$$=-\frac{2}{n\omega}\quad(n\text{ even})\qquad=\frac{8}{n\omega}\quad(n\text{ odd})$$

$$\therefore\ f(t)=\frac{2}{\omega}\left\{4\sin\omega t-\frac{1}{2}\sin 2\omega t+\frac{4}{3}\sin 3\omega t-\frac{1}{4}\sin 4\omega t\ldots\right\}$$

And that just about brings this particular Programme to an end. Fourier series have wide applications so it is very worthwhile paying considerable attention to them.

The **Revision summary** and **Can You?** checklist now follow, after which you will have no trouble with the **Test exercise**. The **Further problems** provide additional practice.

Revision summary 6

86

1 *Graphs of $y=A\sin nx$ and $A\cos nx$*

Amplitude $=A$; period $=\dfrac{360^\circ}{n}=\dfrac{2\pi}{n}$ radians

2 *Harmonics*
$y=A_1\sin x$ is the first harmonic or fundamental
$y=A_n\sin nx$ is the nth harmonic.

3 *Periodic function*

$f(x+P)=f(x)\qquad P=\text{period}$

4 *Fourier series – functions of period 2π*

$$f(x)=\tfrac{1}{2}a_0+a_1\cos x+a_2\cos 2x+a_3\cos 3x+\ldots+a_n\cos nx\ldots$$
$$+b_1\sin x+b_2\sin 2x+b_3\sin 3x+\ldots+b_n\sin nx\ldots$$
$$=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}\{a_n\cos nx+b_n\sin nx\}$$

5 *Dirichlet conditions*

(a) The function $f(x)$ must be defined, single-valued and periodic.
(b) $f(x)$ and $f'(x)$ must be piecewise continuous in the periodic interval.

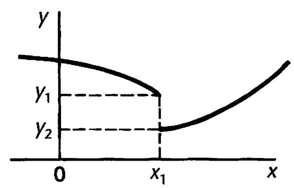
▶

6 *Fourier coefficients*

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

where, in each case, $n = 1, 2, 3, \ldots$

7 *Sum of Fourier series at a finite discontinuity*



At $x = x_1$, series for $f(x)$ converges to the value

$$\tfrac{1}{2}\{f(x_1 - 0) + f(x_1 + 0)\} = \tfrac{1}{2}(y_1 + y_2)$$

8 *Odd and even functions*
(a) Even function: $f(-x) = f(x)$; symmetrical about y-axis.
(b) Odd function: $f(-x) = -f(x)$; symmetrical about the origin.
Product of odd and even functions

(even) × (even) = (even)
(odd) × (odd) = (even)
(odd) × (even) = (odd).

9 *Sine series and cosine series*
If $f(x)$ is *even*, the series contains *cosine terms only* (including a_0).
If $f(x)$ is *odd*, the series contains *sine terms only.*

10 *Half-range series*
Function of period 2π, defined over the range 0 to π. Can be considered as half of an even function, or half of an odd function.

11 *Series containing only odd harmonics or only even harmonics*
If $f(x) = f(x + \pi)$ the Fourier series contains *even harmonics only.*
If $f(x) = -f(x + \pi)$ the Fourier series contains *odd harmonics only.*

12 *Functions with period T*

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos\, n\omega t + b_n \sin\, n\omega t\}$$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,\mathrm{d}t \qquad = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\,\mathrm{d}t$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,\mathrm{d}t \quad = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\cos n\omega t\,\mathrm{d}t$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,\mathrm{d}t \quad = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin n\omega t\,\mathrm{d}t$$

where $\omega = \dfrac{2\pi}{T}$ i.e. $T = \dfrac{2\pi}{\omega}$.

▶

13 *Half-range series – period T*

(a) Even function: half-range cosine series

$$a_0 = \frac{4}{T}\int_0^{T/2} f(t)\,\mathrm{d}t$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\,\mathrm{d}t$$

$$b_n = 0.$$

(b) Odd function: half-range sine series

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin\, n\omega t\,\mathrm{d}t.$$

Can You?

Checklist 6

87

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	**Frames**
• Determine the period and amplitude of a periodic function? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	1 and 2
• Write down the harmonics of a periodic trigonometric function? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	3
• Give an analytic description of a non-sinusoidal periodic function? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	4 to 9
• Evaluate integrals with periodic integrands? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	10 to 15

▶

- Demonstrate the orthogonality of the trigonometric functions $\sin nx$ and $\cos nx$ for $n = 0, 1, 2, \ldots$? **16** to **20**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe a periodic function as a Fourier series subject to the Dirichlet conditions? **21** and **22**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain the Fourier coefficients and hence the Fourier series of a periodic function? **23** to **25**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe the effects of the harmonics in the construction of the Fourier series? **35**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Find the value of the Fourier series at a point of discontinuity of the periodic function? **36** and **37**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive the Fourier series of non-sinusoidal periodic functions? **38** to **44**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise even and odd functions and their products? **45** to **51**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive the Fourier sine and cosine series for odd and even functions respectively? **52** to **64**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive half-range Fourier series? **65** to **73**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise the condition for the Fourier series to contain only odd or only even harmonics? **74** to **78**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Explain the significance of the term $a_0/2$? **79** to **85**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 6

88

1 If $f(x)$ is defined in the interval $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$, state whether or not each of the following functions can be represented by a Fourier series.

(a) $f(x) = x^4$
(b) $f(x) = 3 - 2x$
(c) $f(x) = \dfrac{1}{x}$
(d) $f(x) = e^{2x}$
(e) $f(x) = \operatorname{cosec} x$
(f) $f(x) = \pm\sqrt{4x}$.

2 Determine the Fourier series for the function defined by

$$f(x) = 2x \qquad 0 < x < 2\pi$$
$$f(x+2\pi) = f(x).$$

3 State whether each of the following products is odd, even, or neither.

(a) $x^3 \cos 2x$
(b) $x^2 \sin 3x$
(c) $\sin 2x \sin 3x$
(d) $x^2 e^{2x}$
(e) $(x+5)\cos 2x$
(f) $\sin^2 x \cos x$.

4 A function $f(x)$ is defined by $f(x) = \pi - x \qquad 0 < x < \pi$

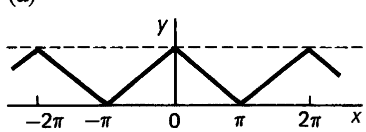
$$f(x+2\pi) = f(x).$$

Express the function

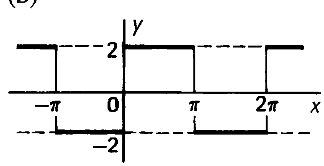
(a) as a half-range cosine series
(b) as a half-range sine series.

5 Comment on the nature of the terms in the Fourier series for the following functions.

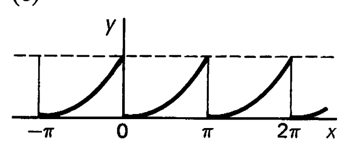
(a)



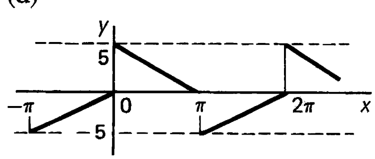
(b)



(c)



(d)



6 A function $f(t)$ is defined by

$$f(t) = \begin{cases} 0 & -2 < t < 0 \\ t & 0 < t < 2 \end{cases}$$
$$f(t+4) = f(t).$$

Determine its Fourier series.

Further problems 6

89

1 A periodic function $f(x)$ is defined by

$$f(x) = 1 - \frac{x}{\pi} \qquad 0 < x < 2\pi$$
$$f(x + 2\pi) = f(x).$$

Determine the Fourier series up to and including the third harmonic.

2 Determine the Fourier series representation of the function $f(t)$ defined by

$$f(t) = \begin{cases} 3 & -2 < t < 0 \\ -5 & 0 < t < 2 \end{cases}$$
$$f(t + 4) = f(t).$$

3 Determine the half-range cosine series for the function $f(x) = \sin x$ defined in the range $0 < x < \pi$.

4 A function is defined by

$$f(x) = \begin{cases} \pi + x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$$
$$f(x + 2\pi) = f(x).$$

Obtain the Fourier series.

5 A periodic function is defined by

$$f(x) = \begin{cases} A\sin x & 0 < x < \pi \\ -A\sin x & \pi < x < 2\pi \end{cases}$$
$$f(x + 2\pi) = f(x).$$

Determine its Fourier series up to and including the fourth harmonic.

6 If $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$ and $f(x + 2\pi) = f(x)$

obtain the Fourier series.

7 Determine the Fourier series to represent a half-wave rectifier output current, i amperes, defined by

$$i = f(t) = \begin{cases} A\sin\omega t & 0 < t < \dfrac{T}{2} \\ 0 & \dfrac{T}{2} < t < T \end{cases}$$
$$f(t + T) = f(t).$$

▶

8 A function $f(x)$ is defined by

$$f(x) = \begin{cases} a & 0 < x < \dfrac{\pi}{3} \\ 0 & \dfrac{\pi}{3} < x < \dfrac{2\pi}{3} \\ -a & \dfrac{2\pi}{3} < x < \pi \end{cases}$$

$f(x+\pi) = f(x)$.

Obtain the Fourier series to represent the function.

9 If $f(x)$ is defined by $f(x) = x(\pi - x) \quad 0 < x < \pi$, express the function as

(a) a half-range cosine series

(b) a half-range sine series.

10 Determine the Fourier cosine series to represent the function $f(x)$ where

$$f(x) = \begin{cases} \cos x & 0 < x < \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} < x < \pi \end{cases}$$

$f(x+2\pi) = f(x)$.

11 If

$$f(x) = \begin{cases} 0 & 0 < x < \dfrac{\pi}{2} \\ \cos x & \dfrac{\pi}{2} < x < \pi \end{cases} \qquad f(x+2\pi) = f(x),$$

obtain the Fourier cosine series for $f(x)$ in the range $x = 0$ to $x = \pi$.

12 A function $f(x)$ is defined over the interval $0 < x < \pi$ by

$$f(x) = \begin{cases} x & 0 < x < \dfrac{\pi}{2} \\ \pi - x & \dfrac{\pi}{2} < x < \pi \end{cases}$$

For the range $x = 0$ to $x = \pi$, determine the Fourier sine series.

13 A function $f(t)$ is defined by

$$f(t) = \begin{cases} -1 & -1 < t < 0 \\ 2t & 0 < t < 1 \end{cases}$$

$f(t+2) = f(t)$.

Obtain the Fourier series up to and including the third harmonic.

14 If $f(x) = x^2 \quad -\pi < x < \pi$ and $f(x+2\pi) = f(x)$, determine the Fourier series for $f(x)$.

15 A function $f(x)$ is given by $f(x) = 7 - \dfrac{3x}{\pi}$ for $-\pi < x < \pi$ with $f(x+2\pi) = f(x)$. Obtain the Fourier series up to and including the fourth harmonic.

▶

16 A function $f(t)$ is defined by

$$f(t) = 1 - t^2 \qquad -1 < t < 1$$
$$f(t+2) = f(t).$$

Determine its Fourier series.

17 A function $f(x)$ is such that

$$f(x) = \begin{cases} \dfrac{x+\pi}{2} & -\pi < x < 0 \\ \dfrac{x-\pi}{2} & 0 < x < \pi \end{cases}$$
$$f(x+2\pi) = f(x).$$

Obtain the Fourier series.

18 Determine the Fourier series for a periodic function such that

$$f(t) = \begin{cases} 1 & -2 < t < -1 \\ 0 & -1 < t < 1 \\ -1 & 1 < t < 2 \end{cases}$$
$$f(t+4) = f(t).$$

19 A function is defined by

$$f(x) = x^2 \qquad 0 < x < 2\pi$$
$$f(x+2\pi) = f(x).$$

Determine its Fourier series.

20 Determine the Fourier series for the function $f(t)$ defined by

$$f(t) = \begin{cases} 0 & -2 < t < 0 \\ \dfrac{3t}{4} & 0 < t < 4 \end{cases}$$
$$f(t+6) = f(t).$$